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Note

Characterization of a Hilbert vector lattice by the metric projection onto its positive cone

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Abstract

If H is a real Hilbert space, K is a closed, generating cone therein and P_K is the metric projection onto K, then the following two conditions 1 and 2 are equivalent:

1. (i) P_K is isotone: $y - x \in K \Rightarrow P_K(y) - P_K(x) \in K$ and (ii) P_K is subadditive: $P_K(x) + P_K(y) - P_K(x+y) \in K, \ \forall x, \ y \in H$, and
2. H ordered by K: (i) is a vector lattice; (ii) $\|x\| = \| \, |x| \, \|, \ \forall x \in H$, and (iii) $x \in K, \ y - x \in K$ imply $\|x\| \leqslant \|y\|$.

Keywords: Sublinear operators

1. Introduction

The positive cone of a Hilbert (vector) lattice (see the definitions in Section 2[1]) is a special form of the isotone projection cone in a real Hilbert space. The simple form of the metric projection onto such a cone, as well as the importance of latticially ordered Hilbert spaces justify their special investigation. In [1,4], there are given characterizations of the Hilbert lattices among the latticially ordered Hilbert spaces.

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[1] The term *Hilbert lattice* is used in the lattice theory also in a different sense. Hence, we pointed out that our definition is referred to as ordered vector spaces in the sense it is used, e.g. in [4].

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This note gives a purely metric projection characterization of a closed generating cone in a Hilbert space (without the a priori assumption of its latticiality) in order to be the positive cone of a Hilbert lattice. (For related questions see [1,5] and the literature therein.)

2. Terminology

Denote by $(H, \langle , \rangle)$ a real Hilbert space. The subset $K \subset H$ is called a *pointed convex cone* if it satisfies the conditions: (i) $K + K \subset K$, (ii) $tK \subset K$, $\forall\ t \in \mathbf{R}_+ = [0, +\infty)$, and (iii)$K \cap -K = \{0\}$. Putting $x \leqslant y$ whenever $y - x \in K$ we get that "$\leqslant$" is a (reflexive, transitive and antisymmetrical) order relation on H, which is translation invariant and invariant with respect to the multiplication with non-negative scalars. Having in mind this order relation, the triplet $(H, \langle , \rangle, K)$ is called an *ordered Hilbert space*, and K is called its *positive cone.* An operator $P : H \to H$ is called *subadditive*, if $P(x) + P(y) - P(x + y) \in K$, $\forall x,\ y \in H$, or using the order relation induced by K, if $P(x + y) \leqslant P(x) + P(y)$, $\forall x,\ y \in H$.

The ordered vector space $(H, \langle , \rangle, K)$ is a *vector lattice* if for every $x,\ y \in H$ there exist $x \wedge y := \inf\{x,\ y\}$ and $x \vee y := \sup\{x,\ y\}$. In this case, we say that K is a *latticial cone* and for each $x \in H$ we define $x^+ = 0 \vee x$, $x^- = 0 \vee (-x)$. Then $x = x^+ - x^-$. Define $|x| = x \vee (-x)$. Then we have $|x| = x^+ + x^-$.

The ordered Hilbert space $(H, \langle , \rangle, K)$ is called a *Hilbert lattice*, if the following conditions hold: (i) $(H, \langle , \rangle, K)$ is a vector lattice, (ii) $||\,|x|\,|| = ||x||$ $\forall x \in H$, and (iii) $0 \leqslant x \leqslant y \Rightarrow ||x|| \leqslant ||y||$.

If K is a cone in H, then the set $K^* = \{y \in H \langle y, x \rangle \geqslant 0\ \forall x \in K\}$ is called the *positive dual cone* of K. It is a closed set and it is a pointed convex cone when K generates H, that is, if $H = K - K$ (this is the case for instance if K is latticial).

If C is a non-empty closed convex set, then for each $x \in H$ there exists a unique element $P_C(x) \in C$ with the property that $||x - P_C(x)||$ =inf$\{||x - y|| : y \in C\}$. The mapping $P_C : H \to C$, thus defined is called the *metric projection* onto C.

The basic tool in our proof will be the following result:

Theorem 1 (Moreau [7]). *If K is a closed cone in H, x, y, $z \in H$, then the following statements are equivalent*:

(i) $z = x + y$, $x \in K$, $y \in -K^*$ *and* $\langle x, y \rangle = 0$.

(ii) $x = P_K(z)$ *and* $y = P_{-K^*}(z)$.

The closed pointed convex cone $K \subset H$ is called an *isotone projection cone*, if $y - x \in K$ implies $P_K(y) - P_K(x) \in K$, or in terms of the order relation induced by K, if $x \leqslant y$ implies $P_K(x) \leqslant P_K(y)$. Another result which we need is the following part of the main result in [3]:

Theorem 2. *If the closed generating pointed convex cone K in H is an isotone projection cone, then it is latticial.*

3. The main result

Theorem 3. *Let $(H, \langle , \rangle)$ be a Hilbert space and let K be a closed generating pointed convex cone therein. Then $(H, \langle , \rangle, K)$ is a Hilbert lattice if and only if P_K is isotone and subadditive with respect to the order induced by K.*

Proof. Some pieces of our proof of the fact that the projection P_K onto the positive cone K of a Hilbert lattice is isotone and subadditive can be obtained in paper [4], or are easy reformulations of the results therein. We prefer to supply the proof here for the sake of completeness.

Let $x, \ y \in K$ with K the positive cone of the Hilbert Lattice. Then

$$|x-y| = (x-y)^+ + (x-y)^- \leqslant x+y,$$

and by properties (ii) and (iii) of the Hilbert lattice it follows that

$$||x-y|| = ||\,|x-y|\,|| \leqslant ||x+y||.$$

From the identity

$$4\langle x, y\rangle = ||x+y||^2 - ||x-y||^2, \tag{1}$$

it follows then that

$$\langle x, y\rangle \geqslant 0 \quad \forall x, \ y \in K. \tag{2}$$

Let x be arbitrary in H. From property (ii) of the Hilbert lattice,

$$||x^+ - x^-|| = ||x^+ + x^-||,$$

which together with (1) yields $\langle x^+, x^- \rangle = 0$.

From relation (2) we have that $K \subset K^*$. This means that $x = x^+ + (-x^-)$, $x^+ \in K$, and $-x^- \in -K \subset -K^*$ with $\langle x^+, -x^- \rangle = 0$; that is, from Theorem 1 it follows that $P_K(x) = x^+$.

Since $x \leqslant y$ implies $x^+ \leqslant y^+$, the isotonicity of P_K follows.

Also, $P_K(x+y) = (x+y)^+ \leqslant x^+ + y^+ = P_K(x) + P_K(y)$, $\forall x, \ y \in H$, and this is the subadditivity of P_K.

Suppose now that K is closed and generating, and P_K is isotone and subadditive with respect to the order induced by K.

By Theorem 2, K is a latticial cone. We have by Moreau's Theorem 1 that $P_K^{-1}(0) = -K^*$. Hence, for each $x \in K$, there holds $0 \leqslant P_K(-x) \leqslant P_K(0) = 0$ and thus $P_K(-K) = \{0\}$. Accordingly, $K \subset K^*$.

Suppose now that P_K is subadditive. Since K is generating, an arbitrary $x \in H$ can be represented as $x = u - v$, with $u, \ v \in K$. Hence, by the subadditivity of P_K,

$$x = u - v = P_K(u) - P_K(v) \leqslant P_K(u-v) = P_K(x)$$

and then $P_K(x) - x \in K$ for each x. Using Moreau's theorem again, we get for $x \in -K^*$ that $-x = P_K(x) - x \in K$ and thus $K^* \subset K$.

If P_K is both isotone and subadditive, then K must be a latticial cone with $K = K^*$. We shall show finally that the relation $K = K^*$ for a latticial cone characterizes the positive cone of a Hilbert lattice (see [4]).

We have only to prove properties (ii) and (iii) of the Hilbert lattice when K is a latticial cone with $K = K^*$. (If $(H, \langle , \rangle, K)$ is a Hilbert lattice we have proved that P_K is isotone and subadditive, and we have just proved that in this case $K = K^*$.)

Let $z \in H$ be arbitrary. Theorem 1 gives

$$-z = P_K(-z) + P_{-K}(-z) \leqslant P_K(-z)$$

and hence $z^- = (-z) \vee 0 \leqslant P_K(-z)$. That is, $P_K(-z) - z^- \geqslant 0$ and from $K = K^*$ and $\langle P_K(-z), P_{-K}(-z) \rangle = 0$, there follows

$$0 \geqslant \langle P_K(-z) - z^-, P_{-K}(-z) \rangle = -\langle z^-, P_{-K}(-z) \rangle.$$

Now $z^- \in K$ and $P_{-K}(-z) \in -K = -K^*$ yield $\langle z^-, P_{-K}(-z) \rangle \leqslant 0$. Hence, $\langle z^-, P_{-K}(-z) \rangle = 0$. But $P_{-K}(-z) = -P_K(z)$ and we have also

$$\langle z^-, P_K(z) \rangle = 0. \tag{3}$$

A similar reasoning yields $P_K(z) - z^+ \geqslant 0$ and from $K = K^*$ and (3) we get

$$0 \leqslant \langle P_K(z) - z^+, z^- \rangle = -\langle z^+, z^- \rangle \Rightarrow \langle z^+, z^- \rangle = 0.$$

Using relation (1) we have then that $||x^+ - x^-|| = ||x^+ + x^-||$, which shows that $||x|| = || \, |x| \, ||$. That is, condition (ii) in the definition of the Hilbert lattice is satisfied.

Suppose that $0 \leqslant x \leqslant y$. From $K = K^*$ we have then

$$0 \leqslant \langle y - x, y \rangle = ||y||^2 - \langle x, y \rangle,$$

and

$$0 \leqslant \langle y - x, x \rangle = \langle y, x \rangle - ||x||^2.$$

Adding we obtain

$$||y||^2 - ||x||^2 \geqslant 0$$

which implies condition (iii). □

4. Remarks

1. An operator Q acting in a vector space is called *positively homogeneous* if for each element x and each non-negative scalar t $Q(tx) = tQ(x)$ holds. Operators in an ordered vector space which are both positively homogeneous and subadditive are called *sublinear*. The metric projection P_K onto a cone K is always positively homogeneous. Hence, in all the definitions and statements above involving the metric projection operator P_K we can write *sublinear* in place of *subadditive*.
2. The relation $x \leqslant P_K(x)$ for each $x \in X$ means that the identity operator is a subgradient of P_K. In this context this result is related to the one in [6].
3. If $H = \mathbf{R}^n$, the n-dimensional Euclidean space, then by the theorem of Youdine every latticial cone is generated by n linearly independent vectors. If any two $n-1$-dimensional faces form an angle $\leqslant \pi/2$, then P_K is isotone [2]. The latticial cones K with $K = K^*$ are the positive orthants of some orthogonal Cartesian

systems. Hence, the cones of this type are the only ones with P_K both isotone and subadditive.

4. The hypothesis of the theorem that K is generating is essential. If K is a one-dimensional cone in $\mathbf{R}^n$, then P_K is both isotone and subadditive. For each cone in R^2 generated by two rays forming a positive acute angle there exist vectors u and v with $P_K(u) + P_K(v) \not\geqslant P_K(u+v)$. This follows from the theorem, but can be verified geometrically that, in this case, there exist elements u and v with $P_K(u) + P_K(v)$ and $P_K(u+v)$ non-comparable. Thus, in R^2 the limit cases: the one-dimensional cone K and the cone K with orthogonal extreme rays are the single ones with P_K both positive and subadditive.

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